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# A LINEAR INVARIANT RELATION IN THE PROBLEM OF THE MOTION OF A GYROSTAT IN A MAGNETIC FIELD†

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The method of invariant relations is used to determine conditions for the existence of a linear invariant relation of the Hess type in a problem of the motion of a heavy rigid body in a magnetic field, taking the Barnett–London effect into account. A special case is indicated in which the reduced system of differential equations of motion has an additional first integral. © 1997 Elsevier Science Ltd. All rights reserved.

In the classical problem of the motion of a heavy rigid body, Hess [1] found a new solution of the Euler–Poisson equations which holds for gyroscopies suspended at some point of an axis passing through the centre of mass and perpendicular to a circular section of the gyration ellipsoid. Sretenskii [2] extended Hess' solution to the case of the motion of a heavy gyrostat. It was then established that an invariant relation of Hess' type exists in other problems of dynamics. Thus, a new solution was found for the Kirchhoff equations of the motion of a body in a fluid, which reduces to Hess' solution under certain conditions [3]; a class of motions was determined for a Hess gyroscope suspended on a rod [4]; and a generalization of Hess's invariant relation was obtained for the problem of the motion of $n$ heavy rigid bodies hinged together [5].

With regard to the problem of the motion of a body in a magnetic field [6, 7], taking the Barnett–London effect into account, some cases have been considered in which new algebraic integrals of motion [8, 9] exist. The study of invariant relations in that problem is therefore of some interest.

## 1. THE EQUATIONS OF MOTION

It is well known that a "neutral" ferromagnetic material, when rotated, becomes magnetized along the axis of rotation (the Barnett effect [6]). An analogous phenomenon is observed when a superconducting solid is rotated (the London effect). The magnetic moment $\mathbf{H}$ is related to the angular velocity $\boldsymbol{\omega}$ by the formula $\mathbf{H} = B\boldsymbol{\omega}$ (the operator $B$ has been calculated for bodies of simple shape [7]).

The equations of motion of a gyrostat in a magnetic field, taking the Barnett–London effect into account, may be written in the form

$$A\dot{\boldsymbol{\omega}} = (\mathbf{A}\boldsymbol{\omega}+\boldsymbol{\lambda})\times\boldsymbol{\omega}+\boldsymbol{\nu}\times(C\boldsymbol{\nu}-\mathbf{s}-B\boldsymbol{\omega}), \quad \dot{\boldsymbol{\nu}}=\boldsymbol{\nu}\times\boldsymbol{\omega} \tag{1.1}$$

These equations admit of first integrals

$$\boldsymbol{\nu}\cdot\boldsymbol{\nu}=1, \quad (A\boldsymbol{\omega}+\boldsymbol{\lambda})\cdot\boldsymbol{\nu}=k \tag{1.2}$$

where $\boldsymbol{\omega}$ is the angular velocity of the gyrostat, $\boldsymbol{\nu}$ is the unit vector in the direction of the gravity field, $\boldsymbol{\lambda}$ is the gyrostatic moment, $\mathbf{s}$ is the vector of the gyrostat's centre of mass, and $A$, $B$ and $C$ are symmetric matrices of order three, $A$ being the inertia tensor of the gyrostat relative to its fixed point.

## 2. HESS-TYPE INVARIANT RELATION

We wish to investigate the conditions under which system (1.1) will admit of an invariant relation

$$\mathbf{x}\cdot\mathbf{s}+\boldsymbol{\nu}\cdot\mathbf{b}=\alpha_0 \tag{2.1}$$

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where $\mathbf{x} = A\boldsymbol{\omega} = (x_1, x_2, x_3)$ is the angular momentum vector of the gyrostat, $\mathbf{b} = (b_1, b_2, b_3)$, and $\alpha_0$ is a constant. We choose the system of coordinates so that the components of the angular velocity have the form

$$\begin{aligned} \omega_l &= a_{ll}x_1 + a_{l3}x_3, \quad l = 1,2 \\ \omega_3 &= a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \end{aligned} \tag{2.2}$$

and so that $\mathbf{s} = (0, 0, 1)$. The numbers $a_{ij}$ in (2.2) are the components of the gyration tensor. It then follows from (2.1) that

$$x_3 = \alpha_0 - \boldsymbol{\nu} \cdot \mathbf{b} \tag{2.3}$$

Let us compute the derivative of (2.3) along trajectories of Eqs (1.1), taking (2.3) into consideration. We obtain

$$\begin{aligned} &(a_{22} - a_{11})x_1x_2 + x_1(a_{23}x_3 - \lambda_2 a_{11}) + x_2(\lambda_1 a_{22} - a_{13}x_3) + x_3(\lambda_1 a_{23} - \lambda_2 a_{13}) + \\ &+x_1(n_{11}\nu_2 - n_{21}\nu_1) + x_2(n_{12}\nu_2 - n_{22}\nu_1) + x_3(n_{13}\nu_2 - n_{23}\nu_1) + \\ &+(C_{22} - C_{11})\nu_1\nu_2 + C_{12}(\nu_1^2 - \nu_2^2) + C_{23}\nu_1\nu_3 - C_{13}\nu_2\nu_3 + \\ &+x_1[a_{11}(b_2\nu_3 - b_3\nu_2) + a_{13}(b_1\nu_2 - b_2\nu_1) + \\ &+x_2[a_{22}(b_3\nu_1 - b_1\nu_3) + a_{23}(b_1\nu_2 - b_2\nu_1) + \\ &+x_3[a_{13}(b_2\nu_3 - b_3\nu_2) + a_{23}(b_3\nu_1 - b_1\nu_3) + a_{33}(b_1\nu_2 - b_2\nu_1)] = 0 \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} &n_{1l} = B_{1l}a_{ll} + B_{13}a_{l3}, \quad n_{l3} = B_{l1}a_{13} + B_{l2}a_{23} + B_{l3}a_{33}, \\ &n_{2l} = B_{l2}a_{ll} + B_{23}a_{l3}, \quad l = 1,2 \end{aligned} \tag{2.5}$$

Relation (2.4) must hold on the invariant manifold (2.3). We therefore suppose that in that relation $x_3 = \alpha_0 - b_1\nu_1 - b_2\nu_2 - b_3\nu_3$. Then the left-hand side of (2.4) is a function of the variables $x_1, x_2, \nu_1, \nu_2, \nu_3$. We require that the left-hand side of (2.4) should vanish for all values of these variables. Then, equating the coefficients of $x_1x_2$ to zero, we obtain

$$a_{22} = a_{11} \tag{2.6}$$

Equality (2.6) enables us to choose a moving system of coordinates so that $a_{23} = 0$. Then the gyrostat's centre of mass will lie on the perpendicular to a circular section of the gyration ellipsoid through the fixed point, that is, the gyrostat will be a Hess gyroscope [1].

Taking (2.5) and (2.6) into account, we obtain two versions of the conditions under which relation (2.4) becomes an identity for all values of $x_1x_2$ and $\nu_1, \nu_2, \nu_3$.

In version 1 we must supplement the relations

$$\begin{aligned} &\alpha_0 = \lambda_1 a_{11}a_{13}^{-1}, \quad \lambda_2 = 0, \quad a_{12} = a_{23} = 0, \quad a_{11} = a_{22} \\ &b_1 = \varkappa_0 a_{13}, \quad b_2 = 0, \quad b_3 = \varkappa_0 a_{11}, \quad B_{12} = B_{23} = 0 \\ &B_{11} = \varepsilon_0 a_{13} + \varkappa_0 a_{11}, \quad B_{22} = \varkappa_0(a_{11}^2 + a_{13}^2)a_{11}^{-1}, \quad B_{13} = -\varepsilon_0 a_{11} - \varkappa_0 a_{13} \\ &C_{12} = C_{23} = 0, \quad C_{13} = \varepsilon_0\varkappa_0 a_{11}(a_{11}a_{33} - a_{13}^2) \\ &C_{11} - C_{22} = \varkappa_0\varepsilon_0 a_{13}(a_{11}a_{33} - a_{13}^2) \end{aligned} \tag{2.7}$$

where $\varepsilon_0$ and $\varkappa_0$ are constant parameters, with a further condition $\lambda_1 = 0$; in version 2 the condition to be added to (2.7) is $\varepsilon_0 = 0$. The invariant relation (2.3) will then be

$$x_3 = \lambda_1 a_{11}a_{13}^{-1} - \varkappa_0(a_{13}\nu_1 + a_{11}\nu_3) \tag{2.8}$$

This relation enables to reduce the order of the system of equations (1.1) and to write a system for the variables $x_1, x_2, \nu_1, \nu_2, \nu_3$

$$\dot{x}_1 = a_{13}x_1x_2 - x_1\nu_2(a_{11}B_{13} + a_{13}B_{33}) + x_2[a_{11}a_{13}^{-1}(-\lambda_2 a_{13} + \lambda_1 a_{33} - \lambda_1 a_{11}) + $$
$$+ a_{13}\varkappa_0(a_{11} - a_{33})\nu_1 + \varkappa_0(a_{13}^2 - a_{11}a_{33} + 2a_{11}^2)\nu_3] + \nu_2\nu_1(C_{13} + \varkappa_0 a_{13}^2 B_{13} + \varkappa_0 a_{13}a_{33}B_{33}) +$$
$$+ \nu_2\nu_3[C_{33} - C_{22} + a_{11}\varkappa_0(a_{13}B_{13} + a_{33}B_{33})] -$$
$$-\nu_2 a_{13}^{-1}[sa_{13} + \lambda_1 a_{11}(B_{13}a_{13} + B_{33}a_{33})] \tag{2.9}$$

$$\dot{x}_2 = -x_1^2 a_{13} + x_1[\lambda_3 a_{11} + \lambda_1 a_{13}^{-1}(a_{11}^2 - a_{13}^2 - a_{11}a_{33}) +$$
$$+ \nu_1(\varkappa_0 a_{13}a_{33} - \varkappa_0 a_{11}a_{13} + a_{13}B_{33} + a_{11}B_{13}) +$$
$$+ \nu_3(\varkappa_0 a_{11}a_{33} - \varkappa_0 a_{11}^2 - a_{13}B_{13} - a_{11}B_{11})] +$$
$$+ \nu_1^2[\varkappa_0 a_{13}(\varkappa_0 a_{13}^2 - a_{13}B_{13} - a_{33}B_{33}) - C_{13}] +$$
$$+ \nu_3^2[\varkappa_0 a_{11}(\varkappa_0 a_{11}a_{13} + a_{13}B_{11} + a_{33}B_{33}) - C_{13}] +$$
$$+ \nu_1\nu_3[\varkappa_0 a_{11}(\varkappa_0 a_{13}^2 - a_{13}B_{13} - a_{33}B_{33}) +$$
$$+ \varkappa_0 a_{13}(\varkappa_0 a_{11}a_{13} + a_{13}B_{11} + a_{33}B_{13}) + C_{11} - C_{33}] +$$
$$+ \nu_1[s + \lambda_1 a_{11}a_{13}^{-1}(a_{13}B_{13} + a_{33}B_{33} - \varkappa_0 a_{13}^2) - \varkappa_0 a_{13}(a_{13}\lambda_3 + \lambda_1(a_{11} - a_{33}))] -$$
$$-\nu_3[\lambda_1 a_{11}a_{13}^{-1}(\varkappa_0 a_{11}a_{13} + a_{13}B_{11} + a_{33}B_{13}) + \varkappa_0 a_{11}(a_{13}\lambda_3 + \lambda_1(a_{11} - a_{33}))] +$$
$$+ \lambda_1 a_{11}a_{13}^{-1}[a_{13}\lambda_3 + \lambda_1(a_{11} - a_{33})] \tag{2.10}$$

$$\dot{\nu}_1 = a_{13}\nu_2 x_1 - a_{11}\nu_3 x_2 + \lambda_1 a_{11}a_{33}a_{13}^{-1}\nu_2 - a_{33}\varkappa_0\nu_2(a_{13}\nu_1 + a_{11}\nu_3) \tag{2.11}$$

$$\dot{\nu}_2 = a_{11}\nu_3 x_1 - a_{13}x_1\nu_1 + \lambda_1 a_{11}\nu_3 - \lambda_1 a_{11}a_{33}a_{13}^{-1}\nu_1 + \varkappa_0(a_{13}\nu_1 + a_{11}\nu_3)(a_{33}\nu_1 - a_{13}\nu_3) \tag{2.12}$$

$$\dot{\nu}_3 = a_{11}(x_2\nu_1 - x_1\nu_2) - \lambda_1 a_{11}\nu_2 + a_{13}\varkappa_0\nu_2(a_{13}\nu_1 + a_{11}\nu_3) \tag{2.13}$$

In all these relations allowance must be made for conditions (2.7) for version 1 or 2.
Equations (2.9)–(2.13) have two integrals

$$(x_1 + \lambda_1)\nu_1 + x_2\nu_2 + \nu_3[(\lambda_3 + \lambda_1 a_{11}a_{13}^{-1}) - \varkappa_0(a_{13}\nu_1 + a_{11}\nu_3)] = \bar{k} \tag{2.14}$$
$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$$

which follow from (1.2) when (2.3) holds.

## 3. A SPECIAL CASE

The above integrals are not enough to reduce the problem to quadratures. We therefore present an example in which the existence of the linear integral (2.8) enables us to reduce the problem to quadratures. Assume that $\varkappa_0 = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $B_{33} = \varepsilon_0 a_{11}^2 a_{13}^{-1}$. Then Eqs (2.9)–(2.13) become

$$\dot{x}_1 = a_{13}x_1x_2 + (C_{33} - C_{22})\nu_2\nu_3 - s\nu_2$$
$$\dot{x}_2 = -a_{13}x_1^2 - (C_{33} - C_{22})\nu_1\nu_2 + s\nu_1$$
$$\dot{\nu}_1 = a_{13}\nu_2 x_1 - a_{11}\nu_3 x_2, \quad \dot{\nu}_2 = x_1(a_{11}\nu_3 - a_{13}\nu_1)$$
$$\dot{\nu}_3 = a_{11}(x_2\nu_1 - x_1\nu_2)$$

which, augmented by the integrals

$$x_1\nu_1 + x_2\nu_2 = k, \quad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1$$

which follow from (2.14), admit of a new integral

$$a_{11}(x_1^2 + x_2^2) + (C_{33} - C_{22})\nu_3^2 - 2s\nu_3 = c$$

When $C_{22} - C_{33} = 0$ we obtain the classical Hess case [1], which has been studied in analytical and geometrical settings by Zhukovskii [10], Nekrasov [11] and Kovalev [12].

## REFERENCES


1. HESS, W., Über die Euler'schen Bewegungsgleichungen und über eine neue partikuläre Lösung des Problems der Bewegung eines starren schweren Körpers um einen festen Punkt. *Math. Ann.*, 1890, **37**, 2, 153–181.
2. SRETENSKII, L. N., Some cases of the integrability of the equations of motion of a gyrostat. *Dokl. Akad. Nauk SSSR*, 1963, **149**, 2, 292–294.
3. KHARLAMOV, P. V., A case of the integrability of the equations of motion of a heavy rigid body in a fluid. *Prikl. Mat. Mekh.*, 1955, **19**, 231–233.
4. GORR, G. V. and KONONYKHIN, G. A., Semi-regular precession of a Hess gyroscope suspended on a rod and its generalization in the problem of the motion of a system of two rigid bodies. *Dokl. Akad. UkrSSR*, Ser. A, No. 2, 48–51, 1987.
5. GORR, G. V. and RUBANOVSKII, V. N., On a new class of motions of a systems of heavy rigid bodies hinged together. *Prikl. Mat. Mekh.*, 1988, **52**, 5, 707–712.
6. KITTEL, C., *Introduction to Solid State Physics*. Wiley, New York, 1962.
7. YEGARMIN, I. Ye., The magnetic field of a rotating superconducting body. In *Aerophysics and Geo-space Research*. Moscow Physico-Technical Institute, Moscow, 1983, pp. 95–96.
8. KOZLOV, V. V., The problem of the rotation of a rigid body in a magnetic field. *Izv. Akad. Nauk SSSR. MTT*, 1985, 6, 28–33.
9. SAMSONOV, V. A., The rotation of a body in a magnetic field. *Izv. Akad. Nauk SSSR.* MTT, 1984, 4, 32–34.
10. ZHUKOVSKII, N. Ye., The loxodromic Hess pendulum. *Trudy Otd. Fiz. Nauk Obshch. Lyubit. Yestestv.*, 1893, **5**, 2, 37–45.
11. NEKRASOV, P. A., Analytical investigation of a case of the motion of a heavy rigid body about a fixed point. *Mat. Sbornik*, 1896, **18**, 2, 161–274.
12. KOVALEV, A. M., The motion of a body in the Hess case. In *Mechanics of Solids*. Naukova Dumka, Kiev, 1, 12–27, 1969.


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